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We have investigated a one-point closure model for compressible turbulence based on third- and higher order cumulant discard for systems undergoing rapid deformation, such as might occur downstream of a shock or other discontinuity. In so doing, we find the lowest order contributions of turbulence to the mean flow, which lead to criteria for Adaptive Mesh Refinement.

I. Introduction†

Rapid distortion theory (RDT) as originally applied by Herring[1] closes the turbulence hierarchy of moment equations by discarding third order and higher cumulants. This is similar to the fourth-order cumulant discard hypothesis of Millionshchikov,[2] except that the Millionshchikov hypothesis was taken to apply to incompressible homogeneous isotropic turbulence generally, whereas RDT is applied only to fluids undergoing a distortion that is "rapid" in the sense that the interaction of the mean flow with the turbulence overwhelms the interaction of the turbulence with itself. It is also similar to Gaussian closure, in which both second and fourth-order cumulants are retained.

Motivated by RDT, we develop a quadratic one-point closure for rapidly distorting compressible turbulence, without regard to homogeneity or isotropy, and make contact with two equation turbulence models, especially the K-ε and K-L models, and with linear instability growth. In the end, we arrive at criteria for Adaptive Mesh Refinement in Finite Volume simulations.

II. Favre Averaged Navier-Stokes Equations

Using a notation in which a comma denotes differentiation, and repeated indices are automatically summed over, we write the Navier-Stokes equations as[3]

$$\partial_t \rho + \left(\rho u_j\right)_{,j} = 0 \tag{1}$$

$$\partial_t\left(\rho u_i\right) + \left(\rho u_i u_j\right)_{,j} - P_{ij,j} = \rho g_i \tag{2}$$

$$\partial_t\left(\rho E\right) + \left(\rho E u_j\right)_{,j} - \left(u_i P_{ij}\right)_{,j} - \left(\kappa T_{,j}\right)_{,j} = \rho u_j g_j \tag{3}$$

Here P_{ij} is the tensor

$$P_{ij} = -p\delta_{ij} + \sigma_{ij} = -p\delta_{ij} + \lambda u_{k,k}\delta_{ij} + \mu\left(u_{i,j} + u_{j,i}\right) \tag{4}$$

μ and λ are the so-called first and second viscosities, which are normally related by

$$\mu + (3/2)\lambda = 0 \tag{5}$$

κ is the thermal conductivity, g_i represents a "gravitational" acceleration, and p is the pressure.

We use Favre (i.e., density-weighted) averages[4] to render the Navier-Stokes equations quadratic in the mean and fluctuating variables, while maintaining explicit conservation form so that algorithms for the resulting closure can be numerically stable.[5]

Favre averages are defined for any quantity w by

$$\rho w = \overline{\rho w} + \rho w' \tag{6}$$

centered such that

$$\overline{\rho w'} = 0 \text{ and } \overline{\rho'} = 0 \tag{7}$$

and normalized so that

$$\tilde{w} = \frac{\overline{\rho w}}{\overline{\rho}} \Rightarrow \overline{\rho w} = \overline{\rho}\tilde{w} \tag{8}$$

Throughout this paper a tilde denotes Favre averaging, a bar denotes Reynolds averaging, and a prime denotes fluctuation with respect to the Favre average. A double prime is used to denote fluctuation with respect to the Reynolds average.

We now use (A2) and discard averages containing 3 or more primed quantities (third and higher-order cumulants) to write the mean flow equations

$$\partial_t \overline{\rho} + \left(\overline{\rho}\tilde{u}_j\right)_{,j} = 0 \tag{9}$$

$$\partial_t\left(\overline{\rho}\tilde{u}_i\right) + \left(\overline{\rho}\tilde{u}_i\tilde{u}_j\right)_{,j} + \left(\overline{\rho}\,\overline{u_i'u_j'}\right)_{,j} - \overline{P}_{ij,j} = \overline{\rho}g_i \tag{10}$$

$$\partial_t\left(\overline{\rho}\tilde{E}\right) + \left(\overline{\rho}\tilde{E}\tilde{u}_j\right)_{,j} + \left(\overline{\rho}\,\overline{E'u_j'}\right)_{,j} - \left(\tilde{u}_i\overline{P}_{ij}\right)_{,j} - \left(\overline{u_i'P_{ij}}\right)_{,j} - \left(\kappa\overline{T}_{,j}\right)_{,j} = \overline{\rho}\tilde{u}_j g_j \tag{11}$$

We can also use (A4) and (A5) to eliminate the Reynolds averages of P and T

$$\partial_t\left(\overline{\rho}\tilde{u}_i\right) + \left(\overline{\rho}\tilde{u}_i\tilde{u}_j\right)_{,j} + \left(\overline{\rho}\,\overline{u_i'u_j'}\right)_{,j} - \tilde{P}_{ij,j} + \frac{\overline{\rho' P'_{ij,j}}}{\overline{\rho}} = \overline{\rho}g_i \tag{10a}$$

$$\begin{aligned}\partial_t\left(\overline{\rho}\tilde{E}\right) + \left(\overline{\rho}\tilde{E}\tilde{u}_j\right)_{,j} + \left(\overline{\rho}\,\overline{E'u_j'}\right)_{,j} - \left(\tilde{u}_i\tilde{P}_{ij}\right)_{,j} - \left(\overline{u_i'P_{ij}'}\right)_{,j} - \left(\kappa\tilde{T}_{,j}\right)_{,j} \\ + \left(\frac{\tilde{u}_i}{\overline{\rho}}\overline{\rho' P'}_{ij}\right)_{,j} + \left(\frac{\tilde{P}_{ij}}{\overline{\rho}}\overline{\rho' u'}_i\right)_{,j} + \left(\frac{\kappa}{\overline{\rho}}\overline{\rho' T'_{,j}}\right)_{,j} = \overline{\rho}\tilde{u}_j g_j\end{aligned} \tag{11a}$$

It is apparent that we must either preserve a distinction between Favre and Reynolds averaged quantities, or carry additional terms in our equations. For now, we choose the former. We see that the main advantage of Favre averages is to preserve the averaged equation of mass conservation, even when the fluid is compressible.

If we add (2) multiplied by u'_j to itself with i and j interchanged, we can obtain after some algebra

$$\partial_t\left(\bar{\rho}\overline{u_i'u_j'}\right)+\left(\tilde{u}_k\bar{\rho}\overline{u_i'u_j'}\right)_{,k}+\tilde{u}_{i,k}\bar{\rho}\overline{u_k'u_j'}+\tilde{u}_{j,k}\bar{\rho}\overline{u_k'u_i'}-\overline{u_i'P_{jk,k}}-\overline{u_j'P_{ik,k}}=0 \tag{12}$$

Similar processes involving (2) and (3) yield

$$\begin{aligned}\partial_t\left(\bar{\rho}\overline{E'u_i'}\right)&+\left(\tilde{u}_j\bar{\rho}\overline{E'u_i'}\right)_{,j}+\tilde{u}_{i,j}\bar{\rho}\overline{E'u_j'}+\tilde{E}_{,j}\bar{\rho}\overline{u_i'u_j'}-\tilde{P}_{kj,j}\overline{u_i'u_k'}-\tilde{P}_{kj}\overline{u_i'u_{k,j}'}-\tilde{u}_{k,j}\overline{u_i'P_{kj}}-\tilde{u}_k\overline{u_i'P_{kj,j}}\\&-\kappa_{,j}\overline{u_i'T_{,j}}-\kappa\overline{u_i'T_{,jj}}-\overline{E'P_{ij,j}}=\bar{\rho}\overline{u_i'u_j'}g_j\end{aligned} \tag{13}$$

$$\begin{aligned}\partial_t\left(\bar{\rho}\overline{E'E'}\right)&+\left(\tilde{u}_j\bar{\rho}\overline{E'E'}\right)_{,j}+2\tilde{E}_{,j}\bar{\rho}\overline{E'u_j'}-2\left[\tilde{P}_{ij,j}\overline{E'u_i'}+\tilde{P}_{ij}\overline{E'u_{i,j}'}+\tilde{u}_{i,j}\overline{E'P_{ij}}+\tilde{u}_i\overline{E'P_{ij,j}}\right]\\&-2\left[\kappa_{,j}\overline{E'T_{,j}}+\kappa\overline{E'T_{,jj}}\right]=2\bar{\rho}\overline{E'u_j'}g_j\end{aligned} \tag{14}$$

We will also need the following three differential equations

$$\partial_t\overline{\rho'\rho'}+\left(\bar{\rho}\overline{\rho'u_j'}\right)_{,j}+\bar{\rho}\overline{\rho'u_{j,j}'}=0 \tag{15}$$

$$\partial_t\left(\bar{\rho}\overline{\rho'u_i'}\right)+\bar{\rho}_{,j}\bar{\rho}\overline{u_i'u_j'}+\bar{\rho}^2\overline{u_i'u_{j,j}'}-\overline{\rho'P_{ij,j}'}=\overline{\rho'\rho'}g_i \tag{16}$$

$$\begin{aligned}\partial_t\left(\bar{\rho}\overline{\rho'E'}\right)&+\bar{\rho}\left(\overline{\rho E'u_j'}\right)_{,j}-\bar{\rho}\overline{\rho E_{,j}'u_j'}-\tilde{P}_{ij,j}\overline{\rho'u_i'}+\tilde{P}_{ij}\overline{\rho'u_{i,j}'}+\tilde{u}_{i,j}\overline{\rho'P_{ij}'}+\tilde{u}_i\overline{\rho'P_{ij,j}'}\\&-\kappa_{,j}\overline{\rho'T_{,j}'}-\kappa\overline{\rho'T_{,jj}'}=\overline{\rho'u_j'}g_j\end{aligned} \tag{17}$$

Equation (15) may be derived by multiplying (1) by ρ and averaging, then subtracting the product of (9) multiplied by $\bar{\rho}$. Similar procedures yield (16) and (17). To see that equations (9)—(17) are closed, we use (4) and (A5) to expand P and its derivatives.

$$\overline{w'P_{ij}}=-\overline{w'p'}\delta_{ij}+\overline{w'u_{k,k}'}\lambda\delta_{ij}+\mu\left(\overline{w'u_{i,j}'}+\overline{w'u_{j,i}'}\right)-\frac{\tilde{P}_{ij}}{\bar{\rho}}\overline{\rho'w'} \tag{18}$$

$$\overline{w'P_{ij,j}}=-\overline{w'p_{,i}'}+\overline{w'u_{j,j}'}\lambda_{,i}+\mu_{,j}\left(\overline{w'u_{i,j}'}+\overline{w'u_{j,i}'}\right)+\overline{w'u_{j,ji}'}(\lambda+\mu)\lambda+\mu\overline{w'u_{i,jj}'}-\frac{\tilde{P}_{ij,j}}{\bar{\rho}}\overline{\rho'w'} \tag{19}$$

Substitution of ρ, E, and u_i for w in equations (18) and (19) completes the closure. Thus, we have a turbulence closure consisting of nine differential equations supplemented by two algebraic relations. The remaining variables, p and T, are related to ρ and E through the material equation of state. We list equations for ideal gases in Appendix B.

III. Two-Equation Models and Instabilities

If we use the standard definition of turbulent kinetic energy per unit mass[6],

$$K = \frac{1}{2}\overline{u'_j u'_j} \tag{20}$$

then from equation (12)

$$\partial_t\left(\overline{\rho}K\right) + \left(\tilde{u}_k\overline{\rho}K\right)_{,k} + \tilde{u}_{j,k}\overline{\rho}\,\overline{u'_k u'_j} - \overline{u'_j P_{jk,k}} = 0 \tag{21a}$$

Using (A5) and a little algebra, this becomes

$$\partial_t\left(\overline{\rho}K\right) + \left(\tilde{u}_k\overline{\rho}K\right)_{,k} + \tilde{u}_{j,k}\overline{\rho}\,\overline{u'_k u'_j} + \overline{u'_k p'_{,k}} - \left(\overline{u'_j\sigma'_{jk}}\right)_{,k} + \frac{\tilde{P}_{jk,k}}{\overline{\rho}}\overline{\rho' u'_j} + \overline{\rho}\varepsilon = 0 \tag{21b}$$

where we have used the standard definition for the turbulent kinetic energy dissipation rate,

$$\overline{\rho}\varepsilon = \overline{u'_{j,k}\sigma'_{jk}} = \lambda\overline{u'_{j,j}u_{k,k}} + \mu\left(\overline{u'_{j,k}u'_{j,k}} + \overline{u'_{j,k}u'_{k,j}}\right) \tag{22}$$

To derive the equation for ε we use (1) to eliminate the convective derivative of ρ from (2, multiply the result by w and Reynolds average, using (10) to eliminate mean flow quantities. Realizing that the terms multiplying w' must sum to zero, we divide by $\overline{\rho}$ and take their gradient with respect to j. which must also be zero. We then replace w' by each of the terms of σ'_{ij} to obtain after some algebra

$$\partial_t\left(\overline{\rho}\varepsilon\right) + 2\overline{\sigma'_{ij}\left[\tilde{u}_k u'_{i'k} + \tilde{u}_{i,k}u'_k - \frac{P_{ik,k}}{\overline{\rho}}\right]_{,j}} = 0 \tag{23a}$$

If we expand terms and eliminate Reynolds averaged quantities using (A5), we obtain

$$\begin{aligned}\partial_t\left(\overline{\rho}\varepsilon\right) &+ 2\left[\tilde{u}_k\overline{\sigma'_{ij}u'_{i'k}} + \tilde{u}_{i,k}\overline{\sigma'_{ij}u'}_k - \frac{\overline{\sigma'_{ij}P'_{ik,k}}}{\overline{\rho}} + \frac{\tilde{P}_{ik,k}}{\overline{\rho}}\overline{\rho'\sigma'_{ij}}\right]_{,j} \\ &-2\left[\tilde{u}_k\overline{\sigma'_{ij,j}u'_{i'k}} + \tilde{u}_{i,k}\overline{\sigma'_{ij,j}u'}_k - \frac{\overline{\sigma'_{ij,j}P'_{ik,k}}}{\overline{\rho}} + \frac{\tilde{P}_{ik,k}}{\overline{\rho}}\overline{\rho'\sigma'_{ij,j}}\right] = 0\end{aligned} \tag{23b}$$

Although there is no "first principles" way to derive a mixing length for a K-L model, we can formally write[7]

$$L = \frac{K^{3/2}}{\varepsilon} \tag{24}$$

Obviously, the equations of this closure do not reduce to those of linear instability theory, because they are explicitly quadratic in the fluctuating variables (see Appendix C). This suggests that the way to initialize our closure is to use the results of linear perturbation theory to set the initial amplitudes and growth rates of the fluctuating velocities. To make this explicit, we could separate the Navier-Stokes equations into mean and fluctuating components, make the usual assumption of a commoving frame, which sets the mean components equal to zero, and solve for the fluctuating components (without averaging) in the incompressible regime. This would simply reproduce the results of classical instability theory, which we would then "plug in" to the equations in section II or III above.

IV. Discussion & Summary

We now have a model for compressible turbulence that is closed, and, unlike two-equation models, retains all the Reynolds stresses. However, it does not retain third-order cumulants, and therefore does not capture the vortex stretching which leads to the Kolmogorov cascade of energy from larger to smaller scales of turbulence, and on to dissipation. Thus we expect this model to be valid only for some period of time (or distance) after a flow undergoes a rapid distortion, such as might be produced by the passage of a shock, before the full turbulent cascade has had time to develop. We therefore suggest that this model might be useful as a "bridge" between models of linear instability growth and fully developed turbulence.

There is, however, another way to use this model. Let us assume that we are using a Finite Volume (FV) scheme in which the mean flow and turbulent quantities are Favre-averaged *within* each grid cell and the time-step[8]. In this case, the quadratic Favre-averaged turbulent quantities are the lowest-order corrections to the mean flow due to lack of grid resolution. Thus one might refine and de-refine an Adaptive Mesh Refinement (AMR) scheme based on whether the averaged turbulent quantities we have introduced (and their growth rates) are negligible compared to the mean flow quantities, for example

$$\left(\bar{\rho}\tilde{u}_i\tilde{u}_j\right)_{,j} >> \left(\bar{\rho}\overline{u_i'u_j'}\right)_{,j} \tag{25}$$

This closure could be implemented in a Direct Numerical Simulation (DNS) using finite volumes in a straightforward (although possibly tedious) manner by including the Favre-averaged fluctuating terms and their evolution equations into the code. Such a code might then be called a quadratic Favre Averaged Navier Stokes (FANS) or a Favre Averaged Finite Volume (FAFV) simulation.

Finally, material strength could be included in this model by modifying P_{ij} to include terms representing stress and plastic strain rate effects.

Appendix A: Favre and Reynolds formulas

Here we collect a few useful formulas regarding Favre and Reynolds averages.

$$\overline{\rho w v} = \overline{\rho(\tilde{w} + w')(\tilde{v} + v')} = \overline{\rho}\tilde{w}\tilde{v} + \overline{\rho}\,\overline{w'v'} \tag{A1}$$

where (7) eliminates "cross terms" between averaged and fluctuating quantities, and the right-most term follows from the hypothesis of third-order cumulant discard. The meaning of this quasilinear approximation has been previously discussed.[9]

Similarly, we can express

$$\begin{aligned}\overline{\rho(\rho w v)_{,j}} &= \overline{\rho w(\rho v)_{,j}} + \overline{\rho w_{,j}\rho v}\\ &= \overline{\left(\overline{\rho w} + \rho w'\right)\left[\left(\overline{\rho v}\right)_{,j} + \left(\rho v'\right)_{,j}\right]} + \overline{\left(\overline{\rho w_{,j}} + \rho w'_{,j}\right)\left(\overline{\rho v} + \rho v'\right)}\\ &= \overline{\rho}\left(\overline{\rho}\tilde{w}\tilde{v}\right)_{,j} + \overline{\rho}\left(\overline{\rho}\,\overline{w'v'}\right)_{,j}\end{aligned} \tag{A2}$$

The Favre averages appear as a consequence of the normalization (8), cross terms are again eliminated by (7), and fluctuations of the density are eliminated by our hypothesis of third-order cumulant discard.

We can eliminate Reynolds averaged quantities and Reynolds averages of single Favre fluctuating quantities by noting that for any quantities w and v,

$$\overline{\rho w'} = 0 = \overline{\left(\overline{\rho} + \rho'\right)w'} \Rightarrow \overline{w'} = -\frac{\overline{\rho' w'}}{\overline{\rho}} \tag{A3}$$

$$\overline{w} = \overline{\tilde{w} + w'} = \tilde{w} + \overline{w'} = \tilde{w} - \frac{\overline{\rho' w'}}{\overline{\rho}} \tag{A4}$$

$$\overline{w'v} = \left(\overline{w'v'} + \frac{\overline{\rho}}{\overline{\rho}}\overline{w'\tilde{v}}\right) = \left(\overline{w'v'} + \frac{\tilde{v}}{\overline{\rho}}\overline{\overline{\rho} w'}\right) = \overline{w'v'} + \frac{\tilde{v}}{\overline{\rho}}\left(\overline{\rho w'} - \overline{\rho' w'}\right) = \overline{w'v'} - \frac{\tilde{v}}{\overline{\rho}}\overline{\rho' w'} \tag{A5}$$

Appendix B: Ideal Gases

For an ideal gas we can write

$$\overline{w'p} = (\gamma - 1)\overline{\rho e' w'} = (\gamma - 1)\left[\overline{\rho E' w'} - \tilde{u}_j\overline{\rho w' u'_j}\right] \tag{B1}$$

$$\overline{w'p_{,i}} = \left[\gamma_{,i} + (\gamma - 1)\overline{\rho}_{,i}\right]\left[\overline{E'w'} - \tilde{u}_j\overline{w'u'_j}\right] + (\gamma - 1)\overline{\rho}\left[\overline{E'_{,i}w'} - \tilde{u}_j\overline{w'u'_{j,i}} - \tilde{u}_{j,i}\overline{w'u'_j}\right] \tag{B2}$$

and

$$\overline{w'T_{,j}} = \overline{w'e_{,j}} - \frac{c_{v,j}}{c_v^2}\overline{w'e} \tag{B3}$$

$$\overline{w'T_{,jj}} = \overline{w'e_{,jj}} - \frac{c_{v,j}}{c_v^2}\overline{w'e_{,j}} + \left[2\frac{\left(c_{v,j}\right)^2}{c_v^3} - \frac{c_{v,jj}}{c_v^2}\right]\overline{w'e} \tag{B4}$$

where

$$\overline{w'e} = \overline{w'E} - \frac{\tilde{u}_j}{2}\left(\overline{w'u_j} + \overline{w'u'_j}\right) \tag{B5}$$

$$\overline{w'e_{,j}} = \overline{w'E_{,j}} - \tilde{u}_{k,j}\overline{w'u'_k} - \tilde{u}_k\overline{w'u_{k,j}} \tag{B6}$$

$$\overline{w'e_{,jj}} = \overline{w'E_{,jj}} - \tilde{u}_{k,jj}\overline{w'u'_k} - \tilde{u}_k\overline{w'u_{k,jj}} - \tilde{u}_{k,j}\left(\overline{w'u_{k,j}} + \overline{w'u'_{k,j}}\right) \tag{B7}$$

The reader should also note that

$$\begin{aligned}\overline{p} &= \left(\gamma-1\right)\left[\overline{\rho}\tilde{E} - \frac{1}{2}\overline{\rho}\tilde{u}_i\tilde{u}_i - \frac{1}{2}\overline{\rho}\overline{u'_iu'_i}\right]\\ \tilde{p} &= \overline{\rho}\left[\overline{p} - \left(\gamma-1\right)\overline{\rho'E'}\right]\end{aligned} \tag{B8}$$

$$c_v\overline{T} \cong \overline{E} - \frac{1}{2}\left(\tilde{u}_i\tilde{u}_i + \overline{u'_iu'_i}\right) + \tilde{u}_i\frac{\overline{\rho'u'_i}}{\overline{\rho}} \tag{B9}$$

Appendix C: Incompressible Flows

In this appendix we assume that the flow is incompressible and thus divergence free. Density fluctuations vanish, which erases the distinction between Favre and Reynolds averages and their fluctuations (see A4). We will retain our notation for continuity, however. Equation (10) reduces to

$$\partial_t\tilde{u}_i + \left[\tilde{u}_i\tilde{u}_j + \overline{u'_iu'_j} + \frac{\tilde{p}}{\overline{\rho}} - gz\delta_{ij}\right]_{,j} = 0 \tag{C1}$$

which can be further simplified by expressing the mean and fluctuating components of $\boldsymbol{u}$ as gradients of potentials and interchanging the order of derivatives.

$$\partial_t \tilde{u}_i + \left[\frac{\tilde{u}^2}{2} + K + \frac{\tilde{p}}{\bar{\rho}} - gz \right]_{,i} = 0 \qquad \text{(C1b)}$$

Similarly, equations (21) and (23) reduce to

$$\partial_t K + \left[\tilde{u}_j \overline{u'_j u'}_k + \frac{\overline{u'_k p'}}{\bar{\rho}} \right]_{,k} = 0 \qquad \text{(C2)}$$

$$\partial_t \varepsilon + \frac{4\mu}{\bar{\rho}} \left[\left(\tilde{u}_k \overline{u'_{i,j} u'_k} \right)_{,i} + \frac{\overline{u'_{i,j} p'_{,i}}}{\bar{\rho}} \right]_{,j} = 0 \qquad \text{(C3)}$$

where we have used

$$\sigma'_{ij} \approx 2\mu u'_{i,j} \qquad \text{(C4)}$$

We also note that

$$\bar{\rho}\varepsilon \approx 2\mu \overline{u'_{i,j} u'_{i,j}} \qquad \text{(C5)}$$

Notes and References

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[1] Herring, J. R., “Investigation of Problems in Thermal Convection,” Journal of the Atmospheric Sciences, **20**, 325-338, July 1963.

[2] Millionshchikov, M. D., “Theory of isotropic homogeneous turbulence,” Dokl. Akad. Nauk SSSR, **32**,No.9, 611-614, and “Theory of isotropic homogeneous turbulence,” Izv. Akad. Nauk SSSR, Ser. Georgr. i Geofiz., **5**,No.4-5, 433-446. A summary is given in Monin, A. S., and A. M. Yaglom, *Statistical Fluid Mechanics*, Vol. 2, MIT Press, Cambridge, MA, 1975, 241-260.

[3] See, for example, Harlow, F. H., and A. A. Amsden, *Fluid Dynamics*, LA-4700, Los Alamos National Laboratory, 1971, 28-30, or any other introductory hydrodynamics text.

[4] For a review see Alexandre J. A. Favre, “Formulation of the Statistical Equations of Turbulent Flows with Variable Density,” in *Studies in Turbulence*, edited by T. B. Gatskii, S Sarkar, and C. G. Speziale, Springer, New York, 1992, pp 324-341.

[5] See R. J. LeVeque, *Numerical Methods for Conservation Laws*, Birkhauser, Boston, 1990, pp 122-129.

[6] Schilling, Oleg, “Single Velocity Two-Equation Turbulence Models,” presented at the Second International Conference on Advanced Computing and Simulation: The Physics of the Rayleigh-Taylor and Richtmeyer-Meshkov Instabilities, Centre for Mathematical Sciences, University of Cambridge, England, 25 June — 6 July 2007.

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[7] Schilling, *ibid.*

[8] FV schemes are known to be quadratic. See for example, Dimitris Drikakis, "Large Eddy Simulation: High-Resolution & High-Order Methods," presented at the Second International Conference on Advanced Computing and Simulation: The Physics of the Rayleigh-Taylor and Richtmeyer-Meshkov Instabilities, Centre for Mathematical Sciences, University of Cambridge, England, 25 June — 6 July 2007. Of course, considering the variables in a DNS to be averaged within each grid cell and time step might appear to erase the distinction between Favre and Reynolds averaging. The distinction is preserved by augmenting the code with the Reynolds averages of pairs of Favre fluctuations (and their evolution equations) derived above, rather than with Reynolds averages of Reynolds fluctuations.

[9] Futterman, J. A. H., and W. P. Dannevik, "Toward a Quasilinear Closure for Compressible Turbulence," Fourth International Workshop on the Physics of Compressible Turbulent Mixing, Cambridge, England, 29 March — 1 April 1993.

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